4.1

$f'(x) = F(x)$ $\quad$ $f(x) = g(x) + c$

$g'(x) = F(x)$ $\quad$ $h(x) = f(x) - g(x)$ assume $h(x)$ is not constant

$h(a) \neq h(b)$

$\frac{h(b) - h(a)}{b-a} \neq 0$

MVT $h'(c) = \frac{h(b) - h(a)}{b-a}$ $\quad$ $h'(x) = F(x) - F(x) = 0$

$\therefore$ assumption is wrong

if two functions have same derivative, only differ by a constant

Antiderivatives/Integrals

$\int 0\,dx = \text{Constant}$

$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C \quad n \neq -1$

$\int [f(x) \pm g(x)]\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$\int k f(x)\,dx = k \int f(x)\,dx$

$\int \sin x = -\cos x + c$

$\int \cos x\,dx = \sin x + c$

$\int \sec^2 x\,dx = \tan x + c$

$\int \csc^2 x\,dx = -\cot x + c$ $\quad$ $\int k\,dx = kx + c$

$\int \sec x \tan x\,dx = \sec x$

$\int \csc x \cot x\,dx = -\csc x + c$

$$\int \frac{1}{x^3}\,dx = \int x^{-3}\,dx = \frac{x^{-2}}{-2} + C = \frac{-1}{2x^2} + C$$

$$\int \tan^2 x\,dx = \int (\sec^2 x - 1)\,dx = \tan x - x + C$$

* Dont forget the C!

$a = -32\ ft/s^2$

$V = \int -32\,dt = -32t + C_1$

$V(0) = C_1 = V_0$

$V = -32t + V_0$

$h = \int (-32t + V_0)\,dt$

$h(t) = -\frac{32t^2}{2} + V_0 t + C_2$

$h(0) = C_2 = h_0$

$h(t) = -16t^2 + V_0 t + h_0$

$$\sum_{i=1}^{N} a_i = a_1 + a_2 + a_3 + \ldots + a_{n-1} + a_n$$

summation

$$\sum_{i=3}^{5} (i^2) = 9 + 16 + 25 = 50$$

$$\sum_{i=1}^{N} i =$$

$$\sum_{i=1}^{N} k = Nk$$

$$\sum_{i=1}^{N} i = 1 + 2 + 3 + \ldots + N-1 + N$$

$S = 1 + 2 + 3 + 4 + \ldots\ N-2 + N-1 + N$

$S = N + N-1 + N-2 \ldots 3 \quad 2 \quad 1$

$2S = (N+1) + (N+1) + (N+1) + \ldots + (N+1)$

$= \boxed{S = \frac{N(N+1)}{2}}$

Prove by induction:

- if P_1 is true
- if P_k is true implies P_{k+1} is true
- then P_N is true for all N

$$\sum_{i=1}^{N} i^2 = \frac{N(N+1)(2N+1)}{6}$$

$$\sum_{i=1}^{1} i^2 = 1^2 = 1 \quad \frac{1(1+1)(2(1)+1)}{6} =$$

- Assume P_k is true

$$\sum_{i=1}^{k+1} \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

CONTINUED →

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2$$

$$\frac{k(k+1)(2k+1)}{6} + \frac{6(k+1)^2}{6}$$

$$= \frac{(k+1)(2k^2+k+6k+6)}{6}$$

$$= \frac{(k+1)(2k^2+7k+6)}{6}$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

$$= \frac{(k+1)(k+1+1)(2(k+1)+1)}{6}$$

the formula

$$\sum_{i=1}^{N} i^3 = \frac{N^2(N+1)^2}{4}$$

$$\sum_{i=1}^{N} (a_i + B_i) = \sum_{i=1}^{N} a_i + \sum_{i=1}^{N} B_i$$

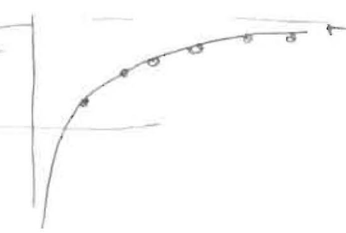
* No product rule

$$\sum_{i=1}^{N} k a_i = k \sum_{i=1}^{N} a$$

$$\lim_{N\to\infty} \sum_{i=1}^{N} a_i$$

if $f(x) = a_N$ for all $x = n$ and $\lim_{x\to\infty} f(x) = L$ then $\lim_{x\to\infty} a_n = L$

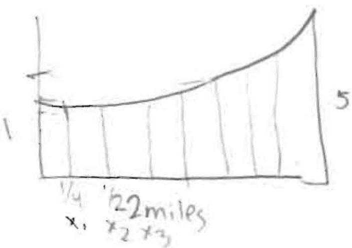
- If limit on series, then function has limit



converse: not true

$f(x) = \sin \pi x$

$a_N = \sin \pi (0)$ ← will always be zero



$y = x^2+1$ $\quad X_0 = 0$ $\quad x_1 = 0+\frac{1}{4}$ $\quad x_2 = 0+2(\frac{1}{4})$

$x_i = \frac{1}{4}i$

$f(\frac{1}{4})(\frac{1}{4}) + f(\frac{1}{2})(\frac{1}{4})$

$x_{i-1} = \frac{1}{4}(i-1)$

OR

$\Delta x = \frac{2-0}{N}$

$x_0 = 0$

$x_1 = 0+\frac{2}{N}$

$x_2 = 0+2(\frac{2}{N})$

$x_i = (\frac{2}{N})i$

$\lim_{N\to\infty} \sum_{i=1}^{N}$ $\longrightarrow$ $\sum_{i=1}^{N} f(\frac{2}{5}i)(\frac{2}{N})$

$\lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{4i^2}{N^2}+1\right)\left(\frac{2}{N}\right)$

- measured from right

$\lim_{N\to\infty} \sum_{i=1}^{n} \left(\frac{8i^2}{N^3} + \frac{2}{N}\right)$

$\lim_{N\to\infty} \frac{8}{N^3}\frac{(N(N+1)(2N+1))}{6} + \frac{2N}{N}$

$= \frac{8}{3} + 2$

$= \frac{14}{3} = 4\frac{2}{3}$

$\lim \sum_{i=1}^{N} (x_{i-1})(\Delta x$

$\lim_{N\to\infty} \sum_{i=1}^{n} f\left(\frac{2}{N}(i-1)\right)\left(\frac{2}{N}\right)$

-measured from left

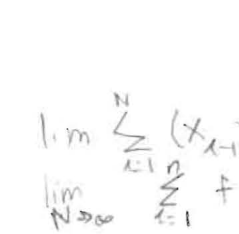
$\lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{4}{N^2}(i^2-2i+1)+1\right)\left(\frac{2}{N}\right)$

$\lim_{N\to\infty} \sum_{i=1}^{N} \left(\frac{8i^2}{N^3} - \frac{16i}{N_3} + \frac{8}{N^3} + \frac{2}{N}\right.$

$\lim_{N\to\infty} \left(\frac{8}{N_3}\left(\frac{N(N+1)(2N+1)}{6}\right) - \frac{16}{N^3}\left(\frac{N(N+1)}{2}\right) + \frac{8}{N^3}N\right.$

$= 8/3 - 0 + 0 + 2$

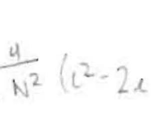
$y = \sin x \; [0, \pi]$



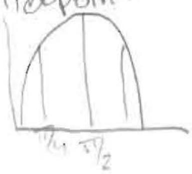
$A_L = 0(\pi/4) + \frac{\sqrt{2}}{2}(\frac{\pi}{4}) + 1(\frac{\pi}{4}) + \frac{\sqrt{2}}{2}(\frac{\pi}{4})$

$= \frac{\pi}{4}(\sqrt{2}+1)$

- right + left balance, but still doesn't mean right answ



Midpoint



$\pi/4\left(\sin\frac{\pi}{8}+\sin\frac{3\pi}{8}+\sin\frac{5\pi}{8}+\sin\frac{7\pi}{8}\right)$

$= 2.05$

Vocab:

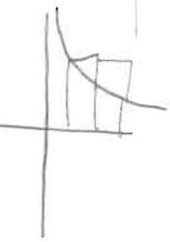
inscribe: inside boxes

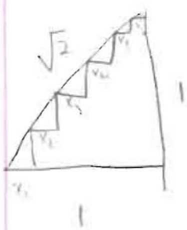
Underaproximation

or

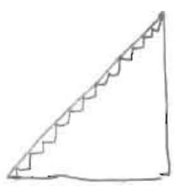
Circumscribe

- over approximation

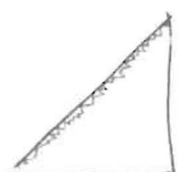




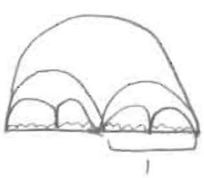
$x_1+x_2+x_3+x_4+x_5=1$



$x_1+x_2+x_3+\dots x_{\text{big}}=1$



$x_1+x_2+x_3+\dots x_k+\dots x_N=\sqrt{2}$



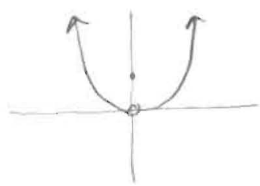
Arclength - π

π 2

$f(x)=\begin{cases}x^2 & \text{if } x\neq 0\\ 3 & \text{if } x=0\end{cases}$

$f'(x)=\begin{cases}2x & \text{if } x\neq 0\\ \text{und.} & \text{if } x=0\end{cases}$

$f'x$

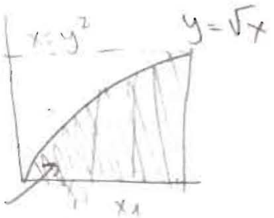


should say increasing $[0,\infty)$ b/c 0 is a critical point, even though not differentiable

$\sum_{K=3}^{100}$

list math sum seq $(Y_1, x, 3, 6)$

- Use Riemann sum on test
- Geometric interpretation circle, rect, triangle, square



$x_0 = 0$

$x_i = 0 + i\Delta x = \frac{i}{n}$

$\Delta x = \frac{1}{n}$

2/3

square is 1x1

$$\lim_{n\to\infty} \sum_{i=1}^{N} f(x_i)\Delta x$$

$$\lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{\frac{i}{N}} \left(\frac{1}{N}\right)$$

$$1 - \lim_{n\to\infty} \sum_{i=y}^{N} g(y_i)\Delta y$$

$y_i = 0 + i\Delta y$

$\Delta y = \frac{1-0}{n} = \frac{1}{n}$

$$1 - \lim_{N\to\infty} \sum_{i=y} \frac{i^2}{N^2}\left(\frac{1}{N}\right)$$

$$1 - \lim_{N\to\infty} \frac{1}{N^3}\left(\frac{N(N+1)(2n+1)}{6}\right)$$

$$1 - \frac{1}{3}$$

$$= \boxed{2/3}$$

TEST QUESTION

* let partition be 1, 3, 8, ...
let $c_i = \frac{1}{2}$, ...
find Riemann sum

Riemann

* can be any size rectangle!

* can measure rectangles from any point

$$\lim_{\|x\|\to 0} \sum_{i=1}^{N} f(c_i)\Delta x_i$$

$x_{i-1} \le c_i \le x_i$

$\Delta x_i = x_i - x_{i-1}$

$\|x\|$ Norm

$\|x\|$ largest Δx_i

$c_i = \frac{i^2}{n^2}$

$c_i = x_i = \frac{i^2}{n^2}$

$\Delta x_i = \frac{i^2}{n^2} - \frac{(i-1)^2}{N^2} = \frac{i^2 - i^2 + \ldots}{N^2}$

$n = 5$

$$\lim_{\|x\|\to 0} \sum_{i=1}^{N} \sqrt{\frac{i^2}{N^2}}\left(\frac{2i-1}{N^2}\right)$$

$$\lim_{n\to\infty} \sum_{i=1}^{N} \left(\frac{2i^2}{N^3} - \frac{i}{N^3}\right)$$

$= \frac{2i-1}{N^2}$

$$\lim_{N\to\infty} \left(\frac{2}{N^3}\left(\frac{N(N+1)(2N+1)}{6}\right) - \frac{1}{N^3}\left(\frac{N(N+1)}{2}\right)\right)$$

$$= \frac{2}{3} - 0$$

$$= \boxed{2/3}$$

$$\lim_{\|\Delta\|\to 0} \sum_{i=1}^{N} f(c_i)\Delta x_i = \int_a^b f(x)\,dx$$

equivalent to $n\to\infty$

Theorem: Integrability implies continuity

* + differentiability implies integrability

$$\int_{-1}^{3} x^2\,dx = \lim_{\|x\|\to 0} \sum_{i=1}^{N} f(c_i)\,\Delta x_i$$

$$\Delta x_i = \Delta x = \frac{3-(-1)}{N} = \frac{4}{N}$$

$$c_i = x_i = -1 + i\,\Delta x = -1 + \frac{4i}{n}$$

$$\lim_{N\to\infty} \left(-1+\frac{4i}{N}\right)^2 \frac{4}{N}$$

etc...

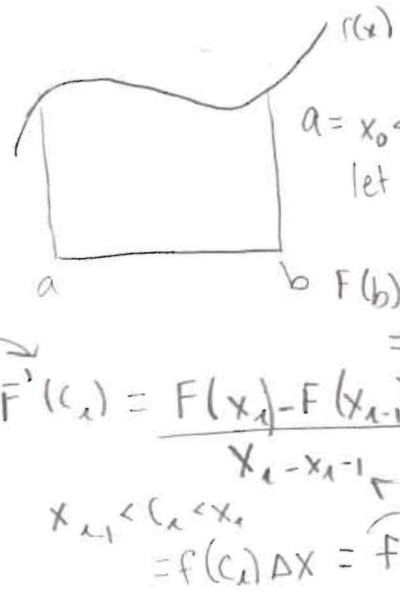
$$\int_0^3 \sqrt{9-x^2}\,dx = \frac{1}{4}\pi\,(3)^2 = \frac{9}{4}\pi$$

4.4. BIGGEST MOMENT IN CALC.

- Fundamental Theory of calculus
- If $F(x)$ is an antiderivative of $f(x)$, then $\int_a^b f(x) = F(b) - F(a)$

$$\int_0^1 \sqrt{x}\,dx = \frac{2}{3}x^{3/2}\Big]_0^1 = \frac{2}{3} - 0 = \frac{2}{3}$$

- it is area if going L to R in a positive function



$a = x_0 < x_1 < x_2 < \dots < x_{i-1} < \dots x_{N-1} < x_N = b$

let $F(x)$ be an antiderivative of $f(x)$ ie $F'(x) = f(x)$

$$F(b) - F(a) = F(x_n) - F(x_{N-1}) + F(x_{N-1}) - F(x_{N-2}) + F(x_{N-2}) \dots F(x_0)$$

$$= \underbrace{\sum_{i=1}^{N} \left(f(x_i) - f(x_{i-1})\right)}_{\text{same}}$$

$f(x) \rightarrow$

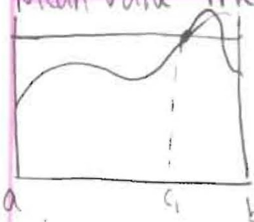
$$F'(c_i) = \frac{F(x_i) - F(x_{i-1})}{\underbrace{x_i - x_{i-1}}_{\Delta x}}$$

$x_{i-1} < c_i < x_i$

$$= f(c_i)\,\Delta x = F(x_i) - F(x_{i-1})$$

$$\therefore \sum_{i=1}^{N} f(c_i)\,\Delta x \qquad \therefore \lim_{\substack{N\to\infty \\ \text{or} \\ \|\Delta\|\to 0}} \sum_{i=1}^{N} f(c_i)\,\Delta x_i \qquad \therefore \int_a^b f(x)\,dx = F_b - F_a$$

Mean Vale Theorem For Integrals



$$\int_a^b f(x)dx = f(c)(b-a)$$

$$\boxed{f(c) = \frac{\int_a^b f(x)dx}{b-(a)}}$$

$$F'(c) = \frac{F(b)-F(a)}{b-a}$$

average value of a function over a certain interval

* if $||\Delta|| \to 0$, then $n \to \infty$
* if $n \to \infty$, $||\Delta||$ does NOT have to go $\to \infty$

$$\int_a^a f(x)dx = 0$$

$$\int_a^b K f(x)dx = K\int_a^b f(x)dx$$

$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

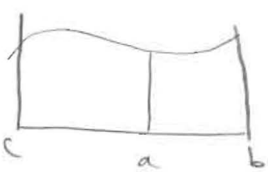
$$\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$$

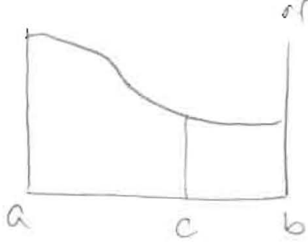
$$\lim_{||\Delta||\to 0} \sum_{i=1}^{N} f(c_i)\Delta x_i \quad \Big| \quad -\lim_{||\Delta||\to 0} \sum_{i=1}^{N} f(c_i)\Delta x_i$$

$$\Delta x = \frac{b-a}{N} \quad \Big| \quad -\Delta x = \frac{a-b}{N}$$

$$\int_a^b f(x)dx = \int_a^c f(x)dx + \int_c^b f(x)dx$$

* True regardless of position of a, b, c





$$\int_c^b f(x)dx = \int_c^a f(x)dx + \int_a^b f(x)dx$$

$$-\int_c^a f(x)dx + \int_c^b f(x)dx = \int_a^b f(x)$$

$$= \int_a^c f(x)dx + \int_c^b f(x)dx =$$

$$\int_a^c f(x)dx + \int_c^B f(x)dx = \int_a^b dx$$

$$\frac{d}{dx}\int_a^x f(t)\,dt = f(x)$$

$$G(x) = \int_0^x f(t)\,dt = F(x) - F(a)$$
$$G'(x) = F'(x) - 0 \qquad F'(x) = f(x)$$

$$g(x) = \int_0^x t^2\,dt = \frac{t^3}{3}\Big]_0^x = \frac{x^3}{3}$$

AP Nice

$$g(x) = \int_3^x \cos(t^2)\,dt$$

$$g'(x) = \cos(x^2)$$

AP Mean

use chain rule

$$g(x) = \int_3^{x^2} \cos t^2\,dt$$

$$g'(x) = \cos(x^2)^2(2x)$$
$$= 2x\cos x^4$$

Rectilinear Motion

$$x(t) = t^3 - 6t^2 + 9t - 2 \qquad 0 \le t \le 5$$
$$V(t) = x'(t) = 3t^2 - 12t + 9$$
$$3(t-1)(t-3)$$
$$v'(t) = a(t) = 6t - 12$$
$$= 6(t-2)$$

t	1	2	3
3	+ +	+	+
$t-1$	− −	+ + +	+
$t-3$	− −	− − −	+
v	+	−	+
a	−		+ +

- particle is at rest when $v=0$
  $t=1 \quad t=3$
- moving right 0-1, 3-5
- when acceleration + v are the same sign, then speed up

$$x(0) = -2$$
$$x(1) = 2 \quad > 4$$
$$x(3) = 27 - 54 + 27 - 2 = -2 \quad > 4$$
$$x(5) = 5(25) - 6(25) + 45 - 2 = 18 \quad > 20$$

$$\boxed{X_T = 28}$$

OR

$\int_0^5 |v(t)|\,dt = \int_0^5 |3t^2 - 12t + 9|\,dt$

$\int_0^1 (3t^2-12t+9)\,dt + \int_1^3 -(3t^2-12t+9)\,dt + \int_3^5 (3t^2-12t+9)\,dt$

* To put in calc: store $3t^2-12t+9$
  Math 9
  fnInt(abs(Y_1), X, 0, 5)

$\int \sin 2x\,dx$ $\quad u = 2x$

$=$ $\quad du = 2\,dx$

$\frac{1}{2}\int 2 \sin 2x\,dx$

$= -\frac{1}{2}\cos u + C$ $\quad u = 2x$

$= \boxed{-\frac{1}{2}\cos 2x + C}$ $\quad du = 2\,dx$

$\int \sin 2x\,dx$

$-\int -\sin x \cos x\,dx = -\int -2\cos x(-\sin x\,dx)$

$u = \cos x$

$du = -\sin x\,dx$

$-\int 2u\,du$

$u^2 + C$

$\boxed{-\cos^2 x + C}$

$\int \sin 2x\,dx$

$\frac{1}{2}\int 2\sin x \cos dx = \boxed{\frac{1}{2}\int 2u\,du}$

$u = \sin x$ $\quad \frac{1}{2}u^2 + C$

$du = \cos x\,dx$

$\frac{1}{2}\sin^2 x + C$

* all answers are correct

$\int x(x^2+1)^{1/2}\,dx$

$\int x\sqrt{x^2+1}\,dx$

$u = x^2+1 \quad du = 2x\,dx$

$\frac{1}{2}\int 2x\sqrt{x^2+1}\,dx$

$= \frac{1}{2}\int \sqrt{u}\,du$

$= \frac{1}{3}\left(\frac{u^{3/2}}{3/2}\right) + C = \frac{1}{3}(x^2+1)^{3/2} + C$

$\int (x^2+1)^{1/2}\,dx$

$\int \sqrt{x^2+1}\,dx$

$\int (x^2+2x+1)^{1/2}\,dx$

$\int \sqrt{(x+1)^2}\,dx$

$\int |x+1|\,dx$

$\int x \sin x^2\,dx$

$u = x^2$

$du = 2x\,dx$

$\frac{1}{2}\int 2x \sin x^2\,dx = \frac{1}{2}\int \sin u\,du = -\frac{1}{2}\cos 2u + C$

$$\int_1^3 \sqrt{2x-1}\,dx$$

$u = 2x-1$

$du = 2\,dx$

$$= \frac{1}{2}\int_1^3 2\sqrt{2x-1}\,dx \qquad 2(1)-1=1$$

$$= \frac{1}{2}\int_1^5 \sqrt{u}\,du \qquad 2(3)-1=5$$

$$= \left(\frac{1}{2}\right)\left(\frac{2}{3}\right)u^{3/2}\Big]_1^5$$

$$= \frac{1}{3}\left(5^{3/2}-1\right)$$